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## LETTER TO THE EDITOR

# Differential formulation of the renormalisation group in the large-$n$ limit

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**Abstract.** A differential formulation of the renormalisation group in the large-$n$ limit is proposed, in terms of which all the standard results can be reproduced in a more natural way. Some advantages of the formulation with respect to the traditional approach are pointed out.

The renormalisation group (RG) approach (Wilson and Kogut 1974) is a potent tool for the study of critical phenomena. However up to the present day, exact non-trivial results, both for classical and quantum systems, have only been obtained in two cases: when the spatial dimensionality $d$ is very close to a critical dimension $d_c$ (Wilson and Kogut 1974, Ma 1976, Young 1975, Gerber and Beck 1977, De Cesare 1978, Busiello and De Cesare 1980a, b, Dacol 1980) and when the order parameter dimensionality $n$ goes to infinity (Ma 1973, 1974, Busiello and De Cesare 1980c). Nevertheless, within the RG approach in the large-$n$ limit, all the results are obtained for large rescaling parameter $b$ and are strictly based on expansions in inverse powers of $b$. Thus, in this case, there is room for more work in exploring some still open areas such as

(i) to solve the RG equations for general $b$;

(ii) to investigate the possibility of multiple solutions;

(iii) to test directly if the expansions used in the standard formulation preserve their validity.

On the other hand, it is well established that a formulation of a RG with a differential generator is far more convenient especially when one manipulates over large domains of the variables involved. With this in mind, we believe that a differential formulation of the RG in the large-$n$ limit could be very useful in clarifying the above points.

In this letter we propose such a formulation for an $n$-vector classical model. It comes out that the RG in the large-$n$ limit can be described by a quasi-linear first-order partial differential equation whose study can be reduced to one of the ordinary differential equations which determine its characteristics. A relevant aspect of the formulation is that the following advantages with respect to the original approach emerge:

($a$) the critical behaviour can be analysed with the same steps used in the usual perturbative approach;

($b$) new explicit results can be obtained in a natural way;

($c$) areas (i)–(iii) can be more easily clarified.

Here we briefly present some preliminary results in support of the proposed formulation. A detailed discussion of the problem, also including an extension to quantum systems, will be the object of a future paper.

As known, for the classical Hamiltonian

$$\mathcal{H} = \int \mathrm{d}^d x[(\nabla\Phi)^2 + U_0(\Phi^2)]$$
$$\Phi \equiv \{\Phi_1, \ldots, \Phi_n\} \qquad U_0(\Phi^2) = \sum_{m=1}^{\infty} \mu_{2m}(\Phi^2)^m \tag{1}$$

the standard RG equations in the large-$n$ limit are (Ma 1973)

$$t'(\Phi^2) = b^2 t_0(b^{2-d}\Phi^2 + \bar{\rho})$$
$$\bar{\rho} = \tfrac{1}{2}nK_d \int_{b^{-1}}^{1} \mathrm{d}\kappa \frac{\kappa^{d-1}}{\kappa^2 + t'/b^2} \tag{2}$$

where $t_0(\Phi^2) = \mathrm{d}U_0/\mathrm{d}\Phi^2$, $K_d = 2^{-(d-1)}\pi^{-d/2}/\Gamma(d/2)$ and a wavevector cut-off equal to unity has been assumed. We now define the infinitesimal RG process $R_{\delta l}$ by writing $b = \mathrm{e}^{\delta l} \simeq 1 + \delta l$, $\delta l \ll 1$, and approximating appropriately the integral involved in equations (2). Successive applications of $R_{\delta l}$ generate a continuous sequence of 'interactions' $t(l, \varphi^2)$ obeying, as can be easily shown, the partial differential equation

$$\frac{\partial t}{\partial l} + (d-2)\left(\varphi^2 - \frac{1}{1+t}\right)\frac{\partial t}{\partial \varphi^2} = 2t \tag{3}$$

to be solved with the initial condition

$$t(0, \varphi^2) = t_0(\varphi^2) = \sum_{m=1}^{\infty} mN_{\mathrm{c}}^{m-1} \mu_{2m}(\varphi^2)^{m-1} \tag{4}$$

where $l$ is a parameter describing the progress of the renormalisation averaging, $\varphi^2 = \Phi^2/N_{\mathrm{c}}$ and $N_{\mathrm{c}} = \frac{1}{2}nK_d/(d-2)$. This equation, whose integration is reduced to a Cauchy problem, constitutes the mentioned differential formulation of the RG in the large-$n$ limit. The fixed points $t^*(\varphi^2)$ of the RG transformation (3), which are defined by the invariance condition $\partial t^*/\partial l = 0$, are determined as solutions of the ordinary first-order differential equation

$$(d-2)\left(\varphi^2 - \frac{1}{1+t^*}\right)\frac{\mathrm{d}t^*}{\mathrm{d}\varphi^2} = 2t^*. \tag{5}$$

It is evident from equation (5) that any solution with $(\mathrm{d}t^*/\mathrm{d}\varphi^2)_{\varphi^2=1} < \infty$ satisfies the relation

$$t^*(\varphi^2)|_{\varphi^2=1} = t^*(1) = 0 \tag{6}$$

for all the dimensionalities $d > 2$. The point $(\varphi^2 = 1, t^* = 0)$ in the plane $(\varphi^2, t^*)$ is a singular point for the fixed point equation (5) and the relation (6) says that any integral curve of equation (5) with finite slope crosses this point.

We immediately see that for any $d > 2$ one has the trivial solution $t^*(\varphi^2) \equiv 0$. Non-trivial fixed points can be obtained by direct integration of equation (5). It is easy to show that a whole one-parameter family of mathematical fixed points exists. Non-trivial physical solutions can be defined by requiring that they preserve the analyticity properties of the original interaction and therefore that they be well behaved everywhere as functions of $\varphi^2$. With this criterion one finds that for $d>2$ there exists a single solution $t^*(\varphi^2)$ which is regular at the point ($\varphi^2=1$, $t^*=0$). Such a solution has, for $2<d<4$, the implicit form

$$\begin{aligned}\varphi^2 &= 1+\frac{d-2}{4-d}\,t^*{}_2F_1(1, 2-d/2; 3-d/2; -t^*)\\ &= 1+\tfrac{1}{2}(d-2)t^*\Phi(-t^*, 1, 2-d/2)\end{aligned} \tag{7}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the usual hypergeometrical function and $\Phi(z, 1, v)$ is the $\Phi$ function whose series representation for $|z|<1$ is $\Phi(z, 1, v)=\Sigma_{m=0}^{\infty} z^m/(m+v)$. Notice that equation (7) is just the implicit representation previously obtained by Ma with a completely different approach (Ma 1974).

In the present formulation it is also easy to obtain an explicit form for the non-trivial physical fixed point as a power expansion around the singular point. Assuming in fact that $t^*(\varphi^2)=\Sigma_{m=0}^{\infty} a_m(\varphi^2-1)^m$, with $a_0=0$ ($\Leftrightarrow t^*(1)=0$), an integration for the series of equation (5) gives for the coefficients $a_m$ the recursion relations

$$\begin{aligned}a_1 &= (4-d)/(d-2)\\ a_m &= \frac{d-2}{d-2(m+1)}\left[\sum_{k=1}^{m-1}\left((m-k)+\frac{2}{d-2}\right)a_k a_{m-k}\right.\\ &\qquad\left.+\sum_{k=1}^{m-2}(m-k)a_{k+1}a_{m-k}\right] \qquad (m\geqslant 2; d>2)\end{aligned} \tag{8}$$

with the convention

$$\left(\sum_{k=1}^{m-2}\cdots\right)_{m=2}=0.$$

To the third order in $(\varphi^2-1)$ we have explicitly

$$\begin{aligned}t^*(\varphi^2) &= \frac{4-d}{d-2}(\varphi^2-1)+\frac{(4-d)^3}{(d-2)^2(6-d)}(\varphi^2-1)^2\\ &\quad+\frac{(4-d)^4[28+d(d-12)]}{(d-2)^3(6-d)^2(8-d)}(\varphi^2-1)^3+\cdots.\end{aligned} \tag{9}$$

From (9) follows to first order in $\varepsilon=4-d$ the well known result (Ma 1974):

$$t^*(\varphi^2)=\tfrac{1}{2}\varepsilon(\varphi^2-1)+O(\varepsilon^2). \tag{10}$$

Of course, from equations (8)–(9), terms of higher order in $\varepsilon$ can be more easily derived than in the original Ma formulation.

The following step consists in establishing the stability of the physical fixed points so that different critical behaviours can be selected. This study can be made by exploring near each fixed point the global solution $t=t(l, \varphi^2)$ of the RG equation (3) which can be determined by means of the well known method of characteristics. However, here we shall give some explicit results based on approximate forms of equation (3) around each fixed point. This is, in some sense, the analogue of the linearisation procedure used in the usual perturbative RG approach.

Very close to a fixed point $t^*(\varphi^2)$, equations (3)–(4) reduce to

$$\frac{\partial\tau}{\partial l}+A^*(\varphi^2)\frac{\partial\tau}{\partial\varphi^2}=B^*(\varphi^2)\tau$$
$$\tau(0,\varphi^2)=\tau_0(\varphi^2)=t_0(\varphi^2)-t^*(\varphi^2) \qquad (11)$$

where $\tau(l,\varphi^2)=t(l,\varphi^2)-t^*(\varphi^2)$, with $|\tau|\ll 1$, and

$$A^*(\varphi^2)=(d-2)\left(\varphi^2-\frac{1}{1+t^*(\varphi^2)}\right)$$
$$B^*(\varphi^2)=2-\frac{d-2}{[1+t^*(\varphi^2)]^2}\frac{\mathrm{d}t^*}{\mathrm{d}\varphi^2}. \qquad (12)$$

For $t^*(\varphi^2)\equiv 0$, we have in (11) $\tau\equiv t$, $A^*=(d-2)(\varphi^2-1)$ and $B^*=2$, and by integration with the method of characteristics, the solution assumes, for $l\gg 1$, the form

$$t(l,\varphi^2)\simeq \mathrm{e}^{\lambda_1 l}t_0(1)+\sum_{m=1}^{\infty}\frac{t_0^{(m)}(1)}{m!}\mathrm{e}^{\lambda_{m+1}l}(\varphi^2-1)^m \qquad (13)$$

where $t_0^{(m)}(1)=(\mathrm{d}^m t_0(x)/\mathrm{d}x)_{x=1}$ and

$$\lambda_1=2 \qquad \lambda_2=4-d$$
$$\lambda_{m+1}=2(m+1)-md=\lambda_2-(m-1)(d-2) \qquad m\geqslant 1. \qquad (14)$$

Since $\lambda_1>0$, $t_0(1)=0$ defines the 'critical surface' on which we have $\lim_{l\to\infty}t(l,\varphi^2)=t^*(\varphi^2)=0$ for $d>4(\Rightarrow\lambda_{m+1}<\ldots<\lambda_2=4-d<0)$. Thus, the trivial fixed point is stable for $d>4$ and the system is characterised by a Gaussian behaviour with $\eta=0$, $\nu=1/\lambda_1=\frac{1}{2}$, $\lambda_2=4-d$, $\lambda_3=6-2d$, $\lambda_4=8-3d,\ldots$. For the non-trivial fixed point, we limit ourselves to the expression $t^*(\varphi^2)=[(4-d)/(d-2)](\varphi^2-1)$ which is valid to the first order in $(\varphi^2-1)$ (or to the first order in $\varepsilon$ with $(4-d)/(d-2)\simeq\frac{1}{2}\varepsilon$). In this case, in equation (11) we have

$$A^*(\varphi^2)=2(\varphi^2-1) \qquad B^*(\varphi^2)=(d-2)+\frac{2(4-d)^2}{d-2}(\varphi^2-1)$$

and for $l\gg 1$ we find

$$\tau(l,\varphi^2)\simeq \mathrm{e}^{\lambda_1 l}\left(1+\frac{(4-d)^2}{d-2}(\varphi^2-1)\right)\tau_0(1)$$
$$+\left(\tau_0'(1)-\frac{(4-d)^2}{d-2}\tau_0(1)\right)\mathrm{e}^{\lambda_2 l}(\varphi^2-1)+\mathrm{O}(\mathrm{e}^{\lambda_3 l}) \qquad (15)$$

where $\lambda_1=d-2>0$, $\lambda_2=d-4$, $\lambda_3=d-6,\ldots$, $\tau_0(1)=t_0(1)-t^*(1)=t_0(1)$ and $\tau_0'(1)=(\mathrm{d}\tau_0(x)/\mathrm{d}x)_{x=1}$. We see that on the critical surface $\tau_0(1)\equiv t_0(1)=0$, we have $\lim_{l\to\infty}\tau(l,\varphi^2)=0$ or $\lim_{l\to\infty}t(l,\varphi^2)=t^*(\varphi^2)$ only for $d<4$. Therefore, the non-Gaussian fixed point is stable only for $d<4$ and the corresponding critical behaviour is of the spherical model type with $\eta=0$, $\nu=1/\lambda_1=1/(d-2)$, $\lambda_2=d-4$, $\lambda_3=d-6,\ldots$. The marginal case $d=4$ is, as usual, more delicate and must be treated with some caution. In this case it is necessary to study the solution of the original RG equation (3) in the limit

$l \to \infty$. It is easy to find for $l \gg 1$

$$t \simeq \frac{e^{2l}}{2u_c l} t_0(1) + (\varphi^2 - 1)/2l \qquad u_c \equiv t_0^{(1)}(1) \neq 0. \tag{16}$$

As we see, on the critical surface, we have

$$t \simeq (\varphi^2 - 1)/2l \xrightarrow{l \to \infty} t^* = 0.$$

Thus, at $d = 4$, the Gaussian fixed point is also stable and the critical behaviour is Gaussian with logarithmic corrections, as expected.

In conclusion, we think that the use of the present differential formulation can be far more convenient for the best comprehension of the structure of the RG in the large-$n$ limit. The main reason is that it allows the use of many techniques familiar from the general theory of differential equations (Garabedian 1964).

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